Numerical bounds for semi-markovian quantities and application to reliability

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Abstract

We propose new easily computable bounds for different quantities which are solutions of Markov renewal equations linked to some continuous-time semi-Markov process (SMP). The idea is to construct two new discrete-time SMP which bound the initial SMP in some sense. The solution of a Markov renewal equation linked to the initial SMP is then shown to be bounded by solutions of Markov renewal equations linked to the two discrete time SMP. Also, the bounds are proved to converge.

To illustrate the results, numerical bounds are provided for two quantities from the reliability field: mean sojourn times and probability transitions.

Key-words: Continuous and discrete time homogeneous semi-Markov processes; Markov renewal equations; Numerical algorithms.

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1 Introduction

Semi-Markov processes (SMP) are used in reliability to model the time-evolution of a system with a finite (or countable) state space divided into up- and down-states. Among quantities of interest to measure the performance of the system are different time-dependent ones, such as: the time-dependent availability, namely the probability that the system is in an up-state at some time t; the mean number of failures of the system on some time interval $[0, t]$; the mean cumulated up- and down-times on $[0, t]$ (see [Csenki (2002)] e.g. for other quantities). If asymptotic quantities are often easily analytically computable for semi-markovian systems, it is usually not the case for time-dependent ones. Indeed, lots of them are solutions of Markov renewal equations, namely Volterra integral equations of the second kind, and only their Laplace transforms are usually analytically reachable. Due to the instability of the numerical inversion of the Laplace transform [Cocozza-Thivent (1997)] or [Csenki (2002)], this is not always a very good method for their numerical computations. Other methods have then been developed such as the phase method, where the idea is to approach an SMP by a Markov process for which computations are much simpler. The main problem here is with the identification of the phases and their number. Other methods consist in the numerical resolution of the Volterra integral equations, see [Csenki (2002)] with lots of reference therein or [Fritz, Posgai & Bertsche (2000)]. An algorithm for the computation of the marginal distribution of an SMP has also been developed recently, using finite volume methods [Cocozza-Thivent & Eymard (2004)]. Other methods consist in approaching continuous time SMP by discrete time SMP for which computations are much simpler (see [Csenki (2002)] with reference therein). In such methods, the main problem is that the accuracy of the results is generally not known.

In the present paper, we propose some method of the last category, namely approaching continuous time SMP by discrete time SMP. The precision of the approximation is here known, and may be made as small as wanted (at least theoretically): for a given continuous-time (homogeneous) SMP, the principle is to construct two new SMP with the same Markov chain as the initial SMP, but with inter-arrival times which bound the initial ones. This allows to bound solutions of Markov renewal equations linked to the initial SMP by solutions of Markov renewal equations linked to the two new SMP. The convergence of the bounds is established with minimal assumptions on the initial semi-Markov kernel (no assumption of density with respect to Lebesgue measure; no need for the inter-arrival times to be almost surely strictly positive; countable but not necessarily finite state space). The two new SMP are discrete time processes and jump only at points kh ($k \in \mathbb{N}$), which allows exact calculation for the resolution of the associated Markov renewal equations as in other methods of the same kind. This provides bounds for different time-dependent quantities linked to some general SMP.

A similar method has been used in [Elkins & Wortman (2001)] to bound the Markov renewal function (a special case of ours) in case of a finite state space, though not described in terms of discrete SMP as here. However, this previous paper is mainly concerned with the numerical computation of the bounds and its implementation, and does not take in consideration the mathematical convergence.

An approximating discrete time SMP has also been used in [Blasi, Janssen & Manca (2004)], [Janssen & Manca (2001)] and [Corradi, Janssen & Manca (2004)], where the almost sure convergence of the approximating discrete time SMP towards the initial continuous-time SMP is proved in the sense of the Skorokhod topology. Contrary to the present paper, the non-homogeneous case is also envisionned in such papers but the initial semi-Markov kernel is assumed to admit density with respect to Lebesgue measure, the state space is finite and the inter-arrival times are assumed to be almost surely positive. Under such assumptions, their approximating discrete time SMP roughly meet with one of those considered here. However, the construction of a second one allows us to get here some bounds for the goal quantities whereas the accuracy of the numerical results is not provided in the quoted papers. Also, the link is not made there between the a.s. convergence in the sense of the Skorokhod topology and the convergence of the appproximation for the goal quantity, the transition probabilities in their case (a special case of ours).

Finally, the same bounding method as in the present paper has already been used in [Mercier (2004)] and [Mercier (2007)] to bounds a few other performance measures from the reliability field, such as 1) cumulative density functions of sums of i.i.d. nonnegative random variables, 2) renewal functions and 3) cumulative density functions of geometric sums of i.i.d. nonnegative random variables. Such performance measures may all be interpreted in terms of SMP so that they can be seen as special cases of the present paper.

The paper is organized as follows: the notations and assumptions are given in Section 2, as well as some recalls on SMP. The two discrete SMP are constructed in Section 3 and bounds for solutions of Markov renewal equations associated to the initial SMP are derived. The convergence of the bounds when the step size h goes to 0 is also proved in this section. Solutions for Markov renewal equations associated to the two new discrete SMP are given in Section 4. Applications and numerical examples are provided in Section 5 and numerical bounds are computed for two quantities of interest for semi-markovian systems: mean sojourn times and transition semi-group. Conclusions are derived in Section 6.

2 Notations, assumptions and recalls

Let E be some finite or countable space and let $(Y_n, T_n)_{n\in\mathbb{N}}$ be a (homogeneous) Markov renewal process with $T_0 = 0$ and $Y_n \in E$ for all $n \in \mathbb{N}$. Also, let $(q(i,j,dt))_{i,j\in E}$ be the associated semi-Markov kernel, namely some family of non-negative measures such that $\sum_{j\in E} q(i,j,dt)$ is a probability measure (all $i \in E$) and such that:

$$\begin{aligned}
&\mathbb{P}(Y_{n+1} = j \cap T_{n+1} - T_n \leq t | Y_0 = i_0, T_1 = t_1, ..., Y_{n-1} = i_{n-1}, T_n = t_n, Y_n = i) \\
&= \mathbb{P}(Y_{n+1} = j \cap T_{n+1} - T_n \leq t | Y_n = i) \\
&= \mathbb{P}(Y_1 = j \cap T_1 \leq t | Y_0 = i) \\
&= \int_{[0,t]} q(i,j,du) \\
&= q(i,j,[0,t]),
\end{aligned}$$

for all $n \in \mathbb{N}$, all $t \geq 0$, all $t_1, .., t_n \in \mathbb{R}_+$, all $i, j \in E$, all $i_0, ..., i_{n-1} \in E$ such that the conditional probability exists.

The transition matrix for the Markov chain $(Y_n)_{n\in\mathbb{N}}$ then is

$$P_{i,j} = \mathbb{P}(Y_1 = j | Y_0 = i) = \mathbb{P}_i(Y_1 = j) = q(i,j,\mathbb{R}_+)$$

for all $i, j \in E$, where $\mathbb{P}_i$ stands for the conditional probability distribution $\mathbb{P}(\cdot | Y_0 = i)$.

We assume that $(T_n)_{n\in\mathbb{N}}$ is such that $\sup_{n\in\mathbb{N}} T_n = +\infty$ a.e. (regularity assumption). We recall from [Cinlar (1975)] that sufficient conditions for that are:

$\mathbb{P}_i\left(T_0 = ... = T_n = ... = 0\right) = 0$ for all $i \in E$ and finite number of transient states for the Markov chain $(Y_n)_{n\in\mathbb{N}}$, or existence of $b, c > 0$ such that $\mathbb{P}_i\left(T_1 > b\right) > c$ for all $i \in E$ (other conditions in [Cinlar (1975)]).

Symbol $(X_t)_{t\geq 0}$ stands for the continuous-time (homogeneous) SMP associated with $(Y_n, T_n)_{n\in\mathbb{N}}$:

$$X_t = Y_n \text{ if } T_n \leq t < T_{n+1},$$

for all $n \in \mathbb{N}$, all $t \geq 0$. The transition semi-group for $(X_t)_{t\geq 0}$ is $(P_t)_{t\geq 0}$ with:

$$P_t\left(i, j\right) = \mathbb{P}_i\left(X_t = j\right),$$

for all $i, j \in E$, all $t \geq 0$. Also, the Markov renewal function $\rho\left(i, j, [0, t]\right)$ associated to $(X_t)_{t\geq 0}$ is

$$\rho\left(i, j, [0, t]\right) = \mathbb{E}_i\left(\sum_{n\geq 0} \mathbf{1}_{\{T_n\leq t\}}\mathbf{1}_{\{Y_n=j\}}\right)$$

for all $i, j \in E$, all $t \geq 0$, where $\mathbf{1}_{\{\}}$ stands for the indicator function. $\rho\left(i, j, [0, t]\right)$ is the mean number of visits to j on $[0; t]$ for $(X_t)_{t\geq 0}$ starting from i. Note that due to assumption $\sup_{n\in\mathbb{N}} T_n = +\infty$ a.e., we have $\mathbb{P}_i\left(T_1 = T_2 = ... = T_n = ... = 0\right) = 0$ for all $i \in E$, so that $\rho\left(i, j, [0, t]\right) < +\infty$ for all $i, j \in E$, all $t \geq 0$, see [Cinlar (1975)]. Finally, the Markov renewal measure $\rho\left(i, j, dt\right)$ is the non-negative Stieltjes measure associated to the non-decreasing right-continuous function $t \longmapsto \rho\left(i, j, [0, t]\right)$.

We now turn to Markov renewal equations and we denote by $\mathbb{B}_+$ the set of all functions $f : E \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that the function $t \longmapsto f\left(i, t\right)$ is uniformly bounded with respect to $i \in E$ on each compact set, namely $\|f\|_{[0,t]} = \sup_{i\in E}\sup_{u\in[0,t]} f\left(i, t\right) < +\infty$ for all $t \geq 0$.

For all $f \in \mathbb{B}_+$, we set:

$$\begin{aligned}(dq * f)\left(i, t\right) &= \sum_{j\in E}\int_{[0,t]} f\left(j, t-u\right) q\left(i, j, du\right) = \sum_{j\in E}\mathbb{E}_i\left(f\left(j, t-T_1\right)\mathbf{1}_{\{T_1\leq t\}}\mathbf{1}_{\{Y_1=j\}}\right)\\ &= \mathbb{E}_i\left(f\left(Y_1, t-T_1\right)\mathbf{1}_{\{T_1\leq t\}}\right)\end{aligned}$$

and

$$\begin{aligned}(d\rho * f)\left(i, t\right) &= \sum_{j\in E}\int_{[0,t]} f\left(j, t-s\right)\rho\left(i, j, ds\right) = \sum_{j\in E}\sum_{n\geq 0}\mathbb{E}_i\left(f\left(j, t-T_n\right)\mathbf{1}_{\{T_n\leq t\}}\mathbf{1}_{\{Y_n=j\}}\right)\\ &= \sum_{n\geq 0}\mathbb{E}_i\left(f\left(Y_n, t-T_n\right)\mathbf{1}_{\{T_n\leq t\}}\right) \qquad (1)\end{aligned}$$

with $dq * f \in \mathbb{B}_+$ and $d\rho * f \in \mathbb{B}_+$.

We recall from [Cinlar (1975)] that, due to the assumption $\sup_{n\in\mathbb{N}} T_n = +\infty$ a.e., for all $g \in \mathbb{B}_+$, the equation

$$f = g + dq * f$$

has one single solution $f_g \in \mathbb{B}_+$ which is:

$$f_g = d\rho * g \tag{2}$$

with $f_g < +\infty$.

3 Bounding f_g

3.1 Construction of $\left(X_t^h\right)_{t\geq 0}$ and of $\left(X_t^{h+}\right)_{t\geq 0}$

Let $\lfloor ... \rfloor$ be the floor function, namely the function from $\mathbb{R}$ to $\mathbb{Z}$ such that, for all $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the single integer such that:

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$$

For any $h > 0$ and any random variable (r.v.) U with general distribution, we may set:

$$U^h = h \left\lfloor \frac{U}{h} \right\rfloor \text{ and } U^{h+} = h \left\lfloor \frac{U}{h} \right\rfloor + h$$

(same notation in all the paper)

with

$$U^h \leq U < U^{h+}$$

and

$$\lim_{h\to 0^+} U^h = \lim_{h\to 0^+} U^{h+} = U$$

namely: $U^h(\omega) \leq U(\omega) < U^{h+}(\omega)$ and $\lim_{h\to 0^+} U^h(\omega) = \lim_{h\to 0^+} U^{h+}(\omega) = U(\omega)$ for all ω in the domain of U.

Now, let $\left(q^h(i,j,dt)\right)_{i,j\in E}$ and $\left(q^{h+}(i,j,dt)\right)_{i,j\in E}$ be the semi-Markov kernels associated to $h > 0$ and $(q(i,j,dt))_{i,j\in E}$, and such that:

$$q^h(i,j,[0,t]) = \mathbb{P}\left(Y_1 = j \cap T_1^h \leq t | Y_0 = i\right) \tag{3}$$

$$q^{h+}(i,j,[0,t]) = \mathbb{P}\left(Y_1 = j \cap T_1^{h+} \leq t | Y_0 = i\right) \tag{4}$$

with $T_1^h = h\left\lfloor \frac{T_1}{h} \right\rfloor$ and $T_1^{h+} = h\left\lfloor \frac{T_1}{h} \right\rfloor + h = T_1^h + h$.

We can then construct two new SMP $\left(X_t^h\right)_{t\geq 0}$ and $\left(X_t^{h+}\right)_{t\geq 0}$ respectively associated to $\left(q^h\left(i,j,dt\right)\right)_{i,j\in E}$ and $\left(q^{h+}\left(i,j,dt\right)\right)_{i,j\in E}$ with the same initial distribution as $(X_t)_{t\geq 0}$. Both of those SMP have semi-Markov kernels supported by $h\mathbb{N}$. In other words, they are discrete-time SMP. The associated Markov chains are "copies" of the initial Markov chain $(Y_n)_{n\in\mathbb{N}}$ associated to $(X_t)_{t\geq 0}$ in the sense that they have the same initial distribution and the same transition matrix, due to

$$q^h\left(i,j,\mathbb{R}_+\right) = q^{h+}\left(i,j,\mathbb{R}_+\right) = q\left(i,j,\mathbb{R}_+\right) = P_{i,j}$$

for all $i,j\in E$. The visited states for $\left(X_t^h\right)_{t\geq 0}$ and $\left(X_t^{h+}\right)_{t\geq 0}$ actually are the same for $\left(X_t^h\right)_{t\geq 0}$ and $\left(X_t^{h+}\right)_{t\geq 0}$ as for the initial SMP $(X_t)_{t\geq 0}$. Those "copies" of $(Y_n)_{n\in\mathbb{N}}$ are still denoted by $(Y_n)_{n\in\mathbb{N}}$ in all the following.

The inter-arrival times for $\left(X_t^h\right)_{t\geq 0}$ and $\left(X_t^{h+}\right)_{t\geq 0}$ respectively are $\left(\left(T_{n+1}-T_n\right)^h\right)_{n\in\mathbb{N}}$ and $\left(\left(T_{n+1}-T_n\right)^{h+}\right)_{n\in\mathbb{N}}$ where

$$\left(T_{n+1}-T_n\right)^h = h\left\lfloor \frac{T_{n+1}-T_n}{h} \right\rfloor \leq T_{n+1}-T_n \leq \left(T_{n+1}-T_n\right)^{h+} = \left(T_{n+1}-T_n\right)^h + h.$$

Then, the SMP $\left(X_t^h\right)_{t\geq 0}$ stays shorter in each state than $(X_t)_{t\geq 0}$ whereas the SMP $\left(X_t^{h+}\right)_{t\geq 0}$ stays longer.

Note that the approximation is made on the inter-arrival times $T_{n+1}-T_n$ and not on the arrival-times T_n. Indeed, in order to construct an approximate SMP, the approximate $n-$th inter-arrival time $\widetilde{T_{n+1}} - \widetilde{T_n}$ given $Y_0 = i_0, \tilde{T}_0 = t_0, ..., Y_{n-1} = i_{n-1}, \tilde{T}_{n-1} = t_{n-1}, Y_n = i, \tilde{T}_n = t_n$ and $Y_{n+1} = j$ should be conditionaly distributed as $\widetilde{T_1}$ given $Y_0 = i$ and $Y_1 = j$. This is true if $\widetilde{T_{n+1}} - \widetilde{T_n}$ depends on T_n and on T_{n+1} only through their increment $T_{n+1}-T_n$ (all $n\in\mathbb{N}$). It is not true any more if the approximation is taken on T_n and T_{n+1} themselves. (For instance, taking $\widetilde{T_n} = h\left\lfloor \frac{T_n}{h} \right\rfloor$ would not lead to a SMP because $h\left\lfloor \frac{T_{n+1}}{h} \right\rfloor - h\left\lfloor \frac{T_n}{h} \right\rfloor \neq h\left\lfloor \frac{T_{n+1}-T_n}{h} \right\rfloor$).

The respective arrival times for $\left(X_t^h\right)_{t\geq 0}$ and $\left(X_t^{h+}\right)_{t\geq 0}$ actually are:

$$T_n^{(h)} = \sum_{i=0}^{n-1}\left(T_{i+1}-T_i\right)^h$$
$$T_n^{(h+)} = \sum_{i=0}^{n-1}\left(T_{i+1}-T_i\right)^{h+} = T_n^{(h)} + nh \tag{5}$$

(note the nh in $T_n^{(h+)}$).

In the case where E is finite and $q(i,j,dt)$ admits some density with respect to Lebesgue measure, $\left(X_t^{h+}\right)_{t\geq 0}$ meets with the approximating discrete time SMP studied in [Blasi, Janssen & Manca (2004)] and [Corradi, Janssen & Manca (2004)] (with approximate arrival times substituted by (5)).

For all other notations than Y_n, $T_n^{(h)}$ and $T_n^{(h+)}$, we add some subscripts ^{h} and $^{h+}$ when referring to the SMP $\left(X_t^h\right)_{t\geq 0}$ and $\left(X_t^{h+}\right)_{t\geq 0}$, respectively. For instance, for $g \in \mathbb{B}_+$, symbol f_g^h stands for the single solution of $f = g + dq^h * f$ (under assumption which ensures existence and singleness, see the following).

We finally express the semi-Markov kernels $\left(q^h(i,j,dt)\right)_{i,j\in E}$ and $\left(q^{h+}(i,j,dt)\right)_{i,j\in E}$ with respect to the initial one $\left(q(i,j,dt)\right)_{i,j\in E}$.

Lemma 1 *For all $i,j \in E$, all $h>0$, the measures $q^h(i,j,dt)$ and $q^{h+}(i,j,dt)$ have supports included in $h\mathbb{N}$ with*

$$\begin{aligned} q^h(i,j,kh) &= q(i,j,[kh,(k+1)h[) \\ q^{h+}(i,j,kh) &= \mathbf{1}_{\{k\geq 1\}} q(i,j,[(k-1)h,kh[) = \mathbf{1}_{\{k\geq 1\}} q^h(i,j,(k-1)h) \end{aligned}$$

for all $k \in \mathbb{N}$.

Proof. It is clear that the supports are included in $h\mathbb{N}$. Besides, for all $k \in \mathbb{N}$, we have:

$$\left(h\left\lfloor \frac{T_1}{h}\right\rfloor \leq kh\right) \Leftrightarrow (T_1 < h(k+1))$$

We get:

$$\begin{aligned} q^h(i,j,kh) &= q^h(i,j,[0,kh]) - \mathbf{1}_{\{k\geq 1\}} q^h(i,j,[0,(k-1)h]) \\ &= q(i,j,[0,(k+1)h[) - \mathbf{1}_{\{k\geq 1\}} q(i,j,[0,kh[) \\ &= q(i,j,[kh,(k+1)h[) \end{aligned}$$

and similar results for q^{h+}. ■

3.2 A technical lemma

We now give a technical condition which ensures us with $\sup_{n\in\mathbb{N}} T_n^{(h)} = +\infty$ a.e. and in particular with existence and uniqueness of solutions to Markov renewal equations associated to $\left(q^h(i,j,dt)\right)_{i,j\in E}$ (clear for $\left(q^{h+}(i,j,dt)\right)_{i,j\in E}$ due to $\sup_{n\in\mathbb{N}} T_n^{(h+)} \geq \sup_{n\in\mathbb{N}} nh = +\infty$, see (5)).

Lemma 2 *If* $C = \sup_{\{(i,j):P_{i,j}\neq 0\}} \mathbb{E}_i\left(\exp\left(-T_1\right)|Y_1 = j\right) < 1$*, then* $\sup_{n\in\mathbb{N}} T_n^{(h)} = +\infty$ *a.e. for all* $0 < h < -\ln C$*. In case* E *is finite, the condition somewhat simpler* $\min_{\{(i,j):P_{i,j}\neq 0\}} \mathbb{E}_i\left(T_1|Y_1 = j\right) > 0$ *is equivalent to the condition* $C < 1$*.*

Proof. For all $n \in \mathbb{N}$ and $t \geq 0$, we have:

$$\begin{aligned}\mathbb{P}\left(T_n^{(h)} \leq t\right) &\leq \mathbb{P}\left(T_n \leq t + nh\right) = \mathbb{P}\left(e^{-(t+nh)} \leq e^{-T_n}\right) \\ &\leq e^{t+nh}\mathbb{E}\left(e^{-T_n}\right)\end{aligned}$$

due to Markov inequality. Besides:

$$\begin{aligned}\mathbb{E}\left(e^{-T_n}\right) &= \mathbb{E}\left(\exp\left(-\sum_{k=0}^{n-1}\left(T_{k+1} - T_k\right)\right)\right) \\ &= \mathbb{E}\left(\mathbb{E}\left(\prod_{k=0}^{n-1}\exp\left(-\left(T_{k+1} - T_k\right)\right)|Y_0, Y_1, ..., Y_n\right)\right) \\ &= \mathbb{E}\left(\prod_{k=0}^{n-1}\mathbb{E}\left(\exp\left(-\left(T_{k+1} - T_k\right)\right)|Y_0, Y_1, ..., Y_n\right)\right)\end{aligned}$$

due to independence of all $T_{k+1} - T_k$ (with $0 \leq k \leq n-1$) given $Y_0, Y_1, ..., Y_n$. By assumption:

$$\mathbb{E}\left(\exp\left(-\left(T_{k+1} - T_k\right)\right)|Y_0, Y_1, ..., Y_n\right) \leq C < 1$$

Whence:

$$\mathbb{P}\left(T_n^{(h)} \leq t\right) \leq e^{t+nh}C^n = e^t e^{n(h+\ln C)} \tag{6}$$

and by monotony:

$$\lim_{n\to+\infty} \downarrow \mathbb{P}\left(T_n^{(h)} \leq t\right) = \mathbb{P}\left(\sup_{n\in\mathbb{N}} T_n^{(h)} \leq t\right) = 0$$

for all $0 < h < -\ln C$ and $t \geq 0$. We derive by monotony again that $\sup_{n\in\mathbb{N}} T_n^{(h)} = +\infty$ a.e. for all $0 < h < -\ln C$. Finally, it is easy to see that, for all $i, j \in E$ such that $P_{i,j} \neq 0$, the condition $\mathbb{E}_i\left(\exp\left(-T_1\right)|Y_1 = j\right) < 1$ is equivalent to $\mathbb{P}_i\left(\exp\left(-T_1\right) < 1|Y_1 = j\right) = \mathbb{P}_i\left(T_1 > 0|Y_1 = j\right) > 0$, which is also equivalent to $\mathbb{E}_i\left(T_1|Y_1 = j\right) > 0$. Whence the result in case E is finite. ■

3.3 Bounds for f_g and convergence of the bounds

We now turn to the main result of this paper which provides us with bounds for f_g and the convergence of the bounds when h goes to 0.

Theorem 3 *Let $(X_t)_{t\geq 0}$ be a semi-Markov process such that* $C = \sup_{\{(i,j):P(i,j)\neq 0\}} \mathbb{E}_i \left(\exp\left(-T_1\right) | Y_1 = j\right) < 1$*. For all $g \in \mathbb{B}_+$:*

1. *if $t \longmapsto g(i,t)$ is non-decreasing for all $i \in E$, then, for all $0 < h < -\ln C$:*

$$f_g^{h+} = d\rho^{h+} * g \leq f_g = d\rho * g \leq f_g^h = d\rho^h * g < +\infty \tag{7}$$

2. *if g is of the shape $g = g_1 - g_2$ with $g_1, g_2 \in \mathbb{B}^+$ and $t \longmapsto g_j(i,t)$ non-decreasing for $j = 1, 2$ (all $i \in E$), then, for all $0 < h < -\ln C$:*

$$f_{g_1}^{h+} - f_{g_2}^h \leq f_g = f_{g_1} - f_{g_2} \leq f_{g_1}^h - f_{g_2}^{h+} < +\infty \tag{8}$$

3. *if g is uniformly continuous on $E \times [0, t]$ where $t \geq 0$ (namely setting $\varepsilon_g(\alpha)$ $= \sup_{j\in E} \sup_{\substack{|x-y|\leq\alpha \\ x,y\in[0,t]}} |g(j,x) - g(j,y)|$, we have $\lim_{\alpha\to 0^+} \varepsilon(\alpha) = 0$), then:*

$$\lim_{h\to 0^+} f_g^h(i,t) = \lim_{h\to 0^+} f_g^{h+}(i,t) = f_g(i,t)$$

Proof. Uniqueness, existence and finiteness of f_g^{h+}, f_g and f_g^h is clear due to Lemma 2 and assumption on $(X_t)_{t\geq 0}$, for all $0 < h < -\ln C$. Besides, inequality (7) may easily be derived from the definition of $d\rho * g$ (see (1)), from the non-increasingness of $u \longmapsto g(j, t-u) \mathbf{1}_{\{u\leq t\}}$ and from the fact that $T_n^{(h)} \leq T_n < T_n^{(h+)}$. This gives the first point. The second point is a direct consequence from the first point using $f_g = f_{g_1} - f_{g_2}$. As for the third point, for $0 < h < -\ln C$ and $n_0 \in \mathbb{N}$ fixed, we may write:

$$\begin{aligned}
&\left|f_g^h(i,t) - f_g(i,t)\right| \\
&= \left|\sum_{n\geq 0} \mathbb{E}_i \left(\mathbf{1}_{\left\{T_n^{(h)}\leq t\right\}} g\left(Y_n, t - T_n^{(h)}\right) - \mathbf{1}_{\{T_n\leq t\}} g\left(Y_n, t - T_n\right)\right)\right| \\
&\leq S_1(n_0, h) + S_2(n_0, h)
\end{aligned}$$

with

$$\begin{aligned}
S_1(n_0,h) &= \sum_{n=0}^{n_0} \mathbb{E}_i \left(\left|\mathbf{1}_{\left\{T_n^{(h)}\leq t\right\}} g\left(Y_n, t - T_n^{(h)}\right) - \mathbf{1}_{\{T_n\leq t\}} g\left(Y_n, t - T_n\right)\right|\right) \\
S_2(n_0,h) &= \sum_{n>n_0} \mathbb{E}_i \left(\left|\mathbf{1}_{\left\{T_n^{(h)}\leq t\right\}} g\left(Y_n, t - T_n^{(h)}\right) - \mathbf{1}_{\{T_n\leq t\}} g\left(Y_n, t - T_n\right)\right|\right)
\end{aligned}$$

Writing

$$
\begin{aligned}
&\left|\mathbf{1}_{\left\{T_n^{(h)}\le t\right\}} g\left(Y_n, t-T_n^{(h)}\right)-\mathbf{1}_{\{T_n\le t\}} g\left(Y_n, t-T_n\right)\right| \\
&=\left|\left(\mathbf{1}_{\left\{T_n^{(h)}\le t\right\}}-\mathbf{1}_{\{T_n\le t\}}\right) g\left(Y_n, t-T_n^{(h)}\right)+\mathbf{1}_{\{T_n\le t\}}\left(g\left(Y_n, t-T_n^{(h)}\right)-g\left(Y_n, t-T_n\right)\right)\right|
\end{aligned}
$$

in $S_1(n_0, h)$, we get

$$
\begin{aligned}
S_1(n_0, h) \le & \sum_{n=0}^{n_0} \mathbb{E}_i\left(\left|\mathbf{1}_{\left\{T_n^{(h)}\le t\right\}}-\mathbf{1}_{\{T_n\le t\}}\right| g\left(Y_n, t-T_n^{(h)}\right)\right) \\
&+\sum_{n=0}^{n_0} \mathbb{E}_i\left(\mathbf{1}_{\{T_n\le t\}}\left|g\left(Y_n, t-T_n^{(h)}\right)-g\left(Y_n, t-T_n\right)\right|\right)
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\left|\mathbf{1}_{\left\{T_n^{(h)}\le t\right\}}-\mathbf{1}_{\{T_n\le t\}}\right| &= \mathbf{1}_{\left\{T_n^{(h)}\le t\right\}}-\mathbf{1}_{\{T_n\le t\}} \le \mathbf{1}_{\{t<T_n<t+nh\}} \\
\text{and } \left|T_n-T_n^{(h)}\right| &\le nh \le n_0 h \text{ for all } 0\le n\le n_0,
\end{aligned}
$$

and using the function $\varepsilon_g(\cdot)$ defined in the theorem and similar argument as for (6), we get:

$$
\begin{aligned}
S_1(n_0, h) &\le \|g\|_{[0,t]} \sum_{n=0}^{n_0} \mathbb{P}_i(t<T_n<t+nh)+\varepsilon_g(n_0 h) \sum_{n=0}^{n_0} \mathbb{P}_i(T_n\le t) \\
&\le \|g\|_{[0,t]} \sum_{n=0}^{n_0} \mathbb{P}_i(t<T_n<t+nh)+(n_0+1)\,\varepsilon_g(n_0 h)
\end{aligned}
$$

with

$$
\lim_{h\to 0^+} \mathbb{P}_i(t<T_n<t+nh)=\mathbb{P}_i(t<T_n\le t)=0
$$

for all $0\le n\le n_0$. We derive:

$$
\lim_{h\to 0^+}\left(S_1(n_0, h)\right)=0
$$

for all fixed n_0. Moreover:

$$
\begin{aligned}
S_2(n_0, h) &\le \|g\|_{[0,t]} \sum_{n>n_0}\left[\mathbb{P}_i\left(T_n^{(h)}\le t\right)+\mathbb{P}_i(T_n\le t)\right] \\
&\le \|g\|_{[0,t]} \sum_{n>n_0}\left[C^n e^{t+nh}+C^n e^t\right] \\
&\le \|g\|_{[0,t]}\left[\frac{e^{h(n_0+1)}}{1-Ce^h}+\frac{1}{1-C}\right] e^t C^{n_0+1}
\end{aligned}
$$

We derive:

$$
\limsup_{h\to 0^+}\left|f_g^h(i,t)-f_g(i,t)\right| \le \|g\|_{[0,t]} \frac{2}{1-C} e^t C^{n_0+1}
$$

for all fixed n_0. Taking the limit when $n_0 \to +\infty$, we get $\limsup_{h\to 0^+} \left| f_g^h(i,t) - f_g(i,t) \right| = 0$ and the result for f_g^h. The result for f_g^{h+} may be proved similarly, which completes the proof. ■

In case where the kernel $(q(i,j,dt))_{i,j\in E}$ is triangular, one can provide alternate bounds in some special cases:

Proposition 4 *Let $(X_t)_{t\geq 0}$ be a semi-Markov process such that $C = \sup_{\{(i,j):P(i,j)\neq 0\}} \mathbb{E}_i(\exp(-T_1)|Y_1 = j) < 1$. Then let $v : E \longrightarrow \mathbb{R}_+$ and let $w : E \times \mathbb{R}_+ \to \mathbb{R}_+$ be such that $w(i,t) = v(i)\,\mathbb{P}_i(T_1 > t)$. Assume E to be equipped with some order relation and the kernel $(q(i,j,dt))_{i,j\in E}$ to be upper triangular. Then if v is non decreasing, we have, for all $0 < h < -\ln(C)$, $t \geq 0$, $i \in E$:*

$$f_{w^{h+}}^{h+}(i,t) \leq f_w(i,t) = \mathbb{E}_i(v(X_t)) \leq f_{w^h}^{h}(i,t) \tag{9}$$

with

$$\begin{aligned} w^h(i,t) &= v(i)\,\mathbb{P}_i\left(T_1^h > t\right) = v(i)\,\mathbb{P}_i\left(T_1 \geq t^{h+}\right) \\ w^{h+}(i,t) &= v(i)\,\mathbb{P}_i\left(T_1^{h+} > t\right) = v(i)\,\mathbb{P}_i\left(T_1 \geq t^{h}\right) \end{aligned}$$

Inequalities are reversed in case $(q(i,j,dt))_{i,j\in E}$ is lower triangular or v is non increasing.

Proof. Setting $\bar{F}_{Y_n}(t) = \mathbb{P}_{Y_n}(T_1 > t) = \mathbb{E}\left(\mathbf{1}_{\{T_{n+1}-T_n > t\}}|Y_n\right)$, we have:

$$\begin{aligned} f_w(i,t) &= \sum_{n\in\mathbb{N}} \mathbb{E}_i\left(\mathbf{1}_{\{T_n\leq t\}} v(Y_n)\,\bar{F}_{Y_n}(t-T_n)\right) \\ &= \sum_{n\in\mathbb{N}} \mathbb{E}_i\left(\mathbf{1}_{\{T_n\leq t\}} v(Y_n)\,\mathbf{1}_{\{T_{n+1}-T_n > t-T_n\}}\right) \\ &= \sum_{n\in\mathbb{N}} \mathbb{E}_i\left(v(Y_n)\,\mathbf{1}_{\{T_n \leq t < T_{n+1}\}}\right) \qquad (10) \\ &= \mathbb{E}_i(v(X_t)) \end{aligned}$$

Beside, starting again from (10):

$$\begin{aligned} f_w(i,t) &= \sum_{n\in\mathbb{N}} \mathbb{E}_i\left(v(Y_n)\left(\mathbf{1}_{\{t<T_{n+1}\}} - \mathbf{1}_{\{t<T_n\}}\right)\right) \\ &= \sum_{n\in\mathbb{N}} \mathbb{E}_i\left(v(Y_n) - v(Y_{n+1})\right)\mathbf{1}_{\{t<T_{n+1}\}} \end{aligned}$$

If $(q(i,j,dt))_{i,j\in E}$ is upper triangular, we know that: $Y_n \leq Y_{n+1}$ and consequently, assuming v to be non decreasing, $v(Y_n) - v(Y_{n+1}) \leq 0$ for all $n \in \mathbb{N}$. Due to $T_n^{(h)} \leq T_n < T_n^{(h+)}$ for all $n \in \mathbb{N}$, we derive:

$$\mathbf{1}_{\left\{t<T_{n+1}^{(h)}\right\}} \leq \mathbf{1}_{\{t<T_{n+1}\}} \leq \mathbf{1}_{\left\{t<T_{n+1}^{(h+)}\right\}}$$

and (9). ■

Remark 5 *Note that* $t \longmapsto w(i,t)$ *is here non increasing whatever the monotony of* v *is, so that* (7) *cannot be applied to* $g = w$*. In the case where* $(q(i,j,dt))_{i,j\in E}$ *is triangular and* v *is monotone,* (9) *provides simpler bounds than* (8)*.*

Remark 6 *In case* $E = \{1;...;m+1\}$ *with only possible transitions* $1 \to 2 \to ... \to m \to m+1$*, taking* $v(i) = \mathbf{1}_{\{m\leq i\}}$ *and* U_i *a random variable with distribution* $q(i,i+1,dt)$ *for* $1 \leq i \leq m$ *(*U_1*, ...,* U_m *independent),* (9) *then provides bounds for* $\mathbb{E}_1(v(X_t)) = \mathbb{P}_1(X_t \geq m) = \mathbb{P}(U_1 + ... + U_m \leq t)$ *which are similar to those obtained in [Mercier (2007)].*

4 Numerical computations

Under assumptions of Theorem 3, it is known that for all $g \in \mathbb{B}_+$ and $0 < h < -\ln C$, the equation $f = g + dq^h * f$ or equivalently

$$f_g^h(i,Nh) = g(i,Nh) + \sum_{j\in E}\sum_{k=0}^{N} q^h(i,j,(N-k)h)\, f_g^h(j,kh) \text{ for all } N \geq 0, \text{ all } i \in E \tag{11}$$

has got one single solution $f_g^h = d\rho^h * g$ (the same for dq^{h+}). We here provide algorithms very easy to emplement for the recursive computation of f_g^h and f_g^{h+} in case E is finite (recursion on N).

Proposition 7 *Let us assume* E *to be finite and* $\min_{\{(i,j):P_{i,j}\neq 0\}} \mathbb{E}_i(T_1|Y_1 = j) > 0$*. Let* $g \in \mathbb{B}_+$ *and* $0 < h < -\ln C$ *where* $C = \max_{\{i,j\in E:P(i,j)\neq 0\}} \mathbb{E}_i(\exp(-T_1)|Y_1 = j) < 1$*. For all* $n \in \mathbb{N}$*, we set:* $q^h(\cdot,\cdot,N) = \left(q^h(i,j,N)\right)_{i,j\in E}$*,* $f_g^h(\cdot,Nh) = \left(f_g^h(i,Nh)\right)_{i\in E}$ *columnwise, the same for* $g(\cdot,Nh)$ *and* $f_g^{h+}(\cdot,Nh)$*. Let* I *be the identity matrix with size cardinal*(E)*. The matrix* $I - q^h(\cdot,\cdot,0)$ *then is non singular and:*

$$f_g^h(\cdot,0) = \left(I - q^h(\cdot,\cdot,0)\right)^{-1} g(\cdot,0)$$
$$f_g^{h+}(\cdot,0) = g(\cdot,0)$$

For all $N \in \mathbb{N}$, we have:

$$f_g^h(\cdot, Nh) = \left(I - q^h(\cdot,\cdot,0)\right)^{-1} \left[g(\cdot, Nh) + \sum_{k=0}^{N-1} q^h(\cdot,\cdot,(N-k)h) f_g^h(\cdot, kh) \right] \tag{12}$$

$$f_g^{h+}(\cdot, Nh) = g(\cdot, Nh) + \sum_{k=0}^{N-1} q^h(\cdot,\cdot,(N-k-1)h) f_g^{h+}(\cdot, kh) \tag{13}$$

Proof. Let λ be a eigen value of $q^h(\cdot,\cdot,0)$. We know that:

$$\begin{aligned} |\lambda| &\leq \max_{i\in E}\left(\sum_{j\in E} q^h(i,j,0)\right) = \max_{i\in E}\left(\sum_{j\in E} \mathbb{P}_i(T_1 < h | Y_1 = j) P(i,j)\right) \\ &= \max_{i\in E}\left(\sum_{j\in E} \mathbb{P}_i\left(e^{-T_1} > e^{-h} | Y_1 = j\right) P(i,j)\right) \leq e^h C < 1 \end{aligned}$$

for all $0 < h < -\ln C$. We derive that 1 is not an eigenvalue of $q^h(\cdot,\cdot,0)$ so that $I - q^h(\cdot,\cdot,0)$ is non singular. The other results may easily be derived from (11) and from $q^{h+}(\cdot,\cdot,N) = \mathbf{1}_{\{N\geq 1\}} q^h(\cdot,\cdot,N-1)$ for all $N \in \mathbb{N}$. ■

Remark 8 *The algorithms provided here are the most natural and might surely be improved (see [Elkins & Wortman (2001)]). As for the computation of $f_g^{h+}(\cdot, Nh)$, a similar algorithm as the present one may also be found in [Barbu, Boussemart, Limnios (2004)] e.g.. As for the computation of $f_g^h(\cdot, Nh)$, the matrix inversion ($\left(I - q^h(\cdot,\cdot,0)\right)^{-1}$) that appears is actually not computed: an LU factorization is provided for $I - q^h(\cdot,\cdot,0)$ once for all at the beginning of the numerical computations and two successive triangular systems are then solved each time quantities like $\left(I - q^h(\cdot,\cdot,0)\right)^{-1}\mathbf{y}$ are computed (with $\mathbf{y}$ a column vector). This saves much computation time.*

5 Applications

We finally provide numerical bounds for two different performance measures from the reliability field: one in the case where $t \longmapsto g(i,t)$ is non-decreasing (the cumulated mean sojourn time), one in the case where $g = g_1 - g_2$ with both g_j non-decreasing with respect of t (the transition probabilities).

In all this section, E is finite, $(X_t)_{t\geq 0}$ is such that $\min_{\{(i,j):P_{i,j}\neq 0\}} \mathbb{E}_i(T_1|Y_1 = j) > 0$ and h is such that $0 < h < -\ln(C) = -\ln\left(\max_{\{i,j\in E:P(i,j)\neq 0\}} \mathbb{E}_i(\exp(-T_1)|Y_1 = j)\right)$ (see Lemma 2).

5.1 Cumulated mean sojourn time

For $i \in E$, $A \subset E$ and $t \geq 0$, we are interested in the cumulated mean sojourn time in A on $[0,t]$ starting from i defined by:

$$C_A(i,t) = \mathbb{E}_i\left(\int_{[0,t]} \mathbf{1}_{\{X_u \in A\}} du\right) = \sum_{j \in A} \int_{[0,t]} P_u(i,j)\, du$$

Setting

$$g_A(i,t) = \mathbf{1}_{\{i \in A\}} \int_{[0,t]} \mathbb{P}_i(T_1 > u)\, du$$

for all $i \in E$, all $t \geq 0$ and using standard methods, one easily gets:

$$C_A = g_A + dq * C_A$$

and hence:

$$C_A = d\rho * g_A$$

where $t \longmapsto g_A(i,t)$ is non-decreasing for all $i \in E$ and where $u \longmapsto g(i,u)$ is uniformly continuous on $[0,t]$. We derive from Theorem 3: $f_{g_A}^{h+} \leq C_A = f_{g_A} \leq f_{g_A}^{h}$ and the convergence of both bounds when $h \to 0^+$. The bounds are then computed with the results of Proposition 7.

Example 9 *A semi-Markov reparable system is considered, which may be up ($i = 1,2,3$) or down ($i = 4,5$) at time t. (The system may go on degrading when down). The system is initially in state 1 and we are interested in the cumulated mean down time on $[0,t]$ starting from 1, namely $C_A(1,t)$ with $A = \{4,5\}$. We take:*

$$q(\cdot,\cdot,dt)$$

$$= \begin{pmatrix} 0 & (1-\gamma)w_1(t)\,dt + \gamma\delta_0 & 0 & 0 & 0 \\ 0 & 0 & (1-\gamma)w_2(t)\,dt + \gamma\delta_0 & 0 & 0 \\ 0 & 0 & 0 & (1-\gamma)w_3(t)\,dt + \gamma\delta_0 & 0 \\ \frac{2}{3}(1-\gamma)w_4(t) & \frac{1}{6}(1-\gamma)w_4(t) & 0 & 0 & \frac{1}{6}(1-\gamma)w_4(t) + \gamma\delta_0 \\ \frac{2}{3}w_5(t) & \frac{1}{6}w_5(t) & 0 & \frac{1}{6}w_5(t) & 0 \end{pmatrix}$$

where δ_0 stands for the Dirac mass at 0 and $w_i(t)$ stands for the Weibull p.d.f.of $W(\alpha_i,\beta_i)$:

$$w_i(t) = \alpha_i \beta_i t^{\beta_i - 1} e^{-\alpha_i t^{\beta_i}} \mathbf{1}_{\mathbb{R}_+}(t)$$

Note that there may be some instantaneous degradation with probability γ so that $q(\cdot,\cdot,dt)$ does not admit density with respect of Lebesgue measure.

We here have:

$$g_A(i,t) = \mathbf{1}_{\{i=4\}}(1-\gamma)\int_{[0,t]} e^{-\alpha_4 t^{\beta_4}}dt + \mathbf{1}_{\{i=5\}}\int_{[0,t]} e^{-\alpha_5 t^{\beta_5}}dt$$

for all $i \in E$, $t \geq 0$ and we take:

$$\alpha_1 = 10^{-6}; \beta_1 = 1.6; \alpha_2 = 10^{-5}; \beta_2 = 2.2; \alpha_3 = 10^{-4}; \beta_3 = 3.3;$$
$$\alpha_4 = 10^{-3}; \beta_4 = 4.5; \alpha_5 = 10^{-3}; \beta_5 = 3.5; \gamma = 10^{-1}$$

which respective means m_i and coefficients of variations cv_i:

$$m_1 \simeq 5.042 \times 10^3; cv_1 \simeq 0.64; m_2 \simeq 1.659 \times 10^2; cv_2 \simeq 0.48;$$
$$m_3 \simeq 14.62; cv_3 \simeq 0.33; m_4 \simeq 4.236; cv_4 \simeq 0.25; m_5 \simeq 6.48; cv_5 \simeq 0.3165$$

The numerical results are displayed in Figure 1 for $h = 6$ as well as the asymptotic direction (see [Cinlar (1975)] e.g.):

$$\begin{aligned} C_A(i,t) &\sim_{t\to+\infty} \sum_{j\in A} \mathbb{E}_j(T_1)\,\rho(1,j,[0,t]) \\ &\sim_{t\to+\infty} t \times \frac{\sum_{j=4}^{5} \mathbb{E}_j(T_1)\,\nu(j)}{\sum_{k=1}^{5} \nu(k)\,\mathbb{E}_k(T_1)} \\ &\simeq 1.49 \times 10^{-3} \times t \end{aligned}$$

where ν is the stationary distribution for the embedded Markov chain $(Y_n)_{n\in\mathbb{N}}$ and $\mathbb{E}_k(T_1) = (1-\gamma)\,m_k$ for $1 \leq k \leq 4$, $\mathbb{E}_4(T_1) = m_5$.

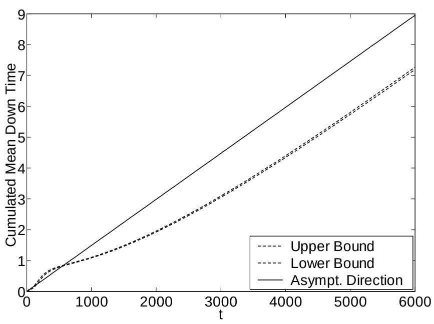

Figure 1. Example 9, Cumulated Mean Down Time on $[0,t]$, $h = 6$.

We can see in such a Figure that the numerical bounds are coherent with the asymptotic direction.

In order to better study the precision of the results, one also provides the results for different values of h and t in Table 1.

t	h	*Lower bound*	*Upper bound*	t	h	*Lower bound*	*Upper bound*
	20	0.4929	0.6487		20	2.2879	2.4144
	12	0.5431	0.6344		12	2.3215	2.3977
300	6	0.5803	0.6251	2400	6	2.3480	2.3863
	3	0.5949	0.6171		3	2.3585	2.3776
	1	0.6048	0.6121		1	2.3658	2.3722
	20	0.8088	0.8856		20	3.3479	3.5063
	12	0.8325	0.8773		12	3.3899	3.4853
600	6	0.8496	0.8716	3300	6	3.4229	3.4708
	3	0.8560	0.8670		3	3.4359	3.4598
	1	0.8605	0.8641		1	3.4450	3.4530
	20	0.9928	1.0585		20	4.5284	4.7165
	12	1.0111	1.0503		12	4.5781	4.6913
900	6	1.0251	1.0447	4200	6	4.6171	4.6739
	3	1.0306	1.0404		3	4.6323	4.6607
	1	1.0344	1.0376		1	4.6431	4.6526
	20	1.1894	1.2651		20	5.7825	5.9991
	12	1.2095	1.2552		12	5.8395	5.9699
1200	6	1.2255	1.2484	5100	6	5.8842	5.9495
	3	1.2318	1.2433		3	5.9015	5.9342
	1	1.2363	1.2401		1	5.9139	5.9248
	20	1.4195	1.5087		20	7.0787	7.3236
	12	1.4430	1.4969		12	7.1429	7.2903
1500	6	1.4618	1.4889	6000	6	7.1931	7.2670
	3	1.4693	1.4828		3	7.2124	7.2494
	1	1.4745	1.4790		1	7.2260	7.2387

Table 1. Example 9: Cumulated Mean Down Time with respect of h and t

We can see in such a table that, as expected, the smaller the discretization step h, the tighter the bounds are. Also, for h fixed, the error first decreases when t increases (for $t \lesssim 900$) and then increases with t. For small t, one may think that the lack of precision is due to some round-off error. For bigger t, the loss of precision when t increases is clearly due to the method.

5.2 Transition semi-group

We are now interested in computing the transiton semi-group $P_t(i,j)$ and we set: $f_j(i,t) = P_t(i,j)$ for $i, j \in E$ and $t \geq 0$. We recall that, setting $g_j(i,t) = 1_{\{i=j\}}\mathbb{P}_i(T_1 > t)$ for all $i, j \in E$, all $t \geq 0$, we have:

$$f_j = g_j + dq * f_j$$

namely $f_j = f_{g_j} = d\rho * g_j$, see [Cocozza-Thivent (1997)] e.g. or [Cinlar (1975)]. Noting that $g_j = I_j - u_j$ with $I_j(i,t) = \mathbf{1}_{\{i=j\}}$ and $u_j(i,t) = 1_{\{i=j\}}\mathbb{P}_i(T_1 \leq t)$, we get $P_t(i,j) = f_{g_j}(i,t) = f_{I_j}(i,t) - f_{u_j}(i,t)$ with $f_{I_j}(i,t) = \rho(i,j,[0,t])$. As I_j and u_j are both non-decreasing with respect of t for all $j \in E$, we derive from (8) :

$$\rho^{h+}(i,j,[0,t]) - f^h_{u_j}(i,t) \leq P_t(i,j) = f_{g_j}(i,t) \leq \rho^h(i,j,[0,t]) - f^{h+}_{u_j}(i,t) \quad (14)$$

and the convergence of both bounds in case $t \longmapsto \mathbb{P}_i(T_1 \leq t)$ is continuous.

The computation of the numerical bounds provided by (14) is here much longer than in the previous case where $t \longmapsto g_j(i,t)$ were non-decreasing. This has lead us to adapt the program in order to take into account only non trivial terms in $(12 - 13)$. Indeed, in reliability theory, mean down times are often much shorter than mean up times so that non trivial terms (at a given precision) among the $q^h(i,j,kh)$'s may highly depend on i and j. Keeping only the non trivial terms in $\sum_{k=0}^{N-1} q^h(i,j,(N-k)h) f^h_g(j,kh)$ then save much computation time.

Example 10 *We here consider an example extracted from [Cocozza-Thivent & Eymard (to appear)] denoted by [CTE] in the following: a system is considered with two components in cold stand-by redundancy (see [CTE] for details) and its evolution is described* *by a SMP with state space* $E = \{1, 2, 3, 4\}$ *and semi-Markov kernel:*

$$q(\cdot,\cdot,dt) = \begin{pmatrix} 0 & w_1(t)\bar{W}_1'(t) & 0 & w_1'(t)\bar{W}_1(t) \\ l_2(t)\bar{W}_2'(t) & 0 & w_2'(t)\bar{L}_2(t) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} dt$$

where $\bar{W}_i(t)$ *and* $\bar{L}_i(t)$ *stand for the respective survival functions of* $W(\alpha_i, \beta_i)$ *and of log-normal distribution* $LN(\alpha_i, \beta_i)$ *with p.d.f.*

$$l_{\alpha_i,\beta_i}(t) = \frac{1}{\beta t\sqrt{2\pi}} e^{-\frac{(\ln(t)-\alpha_i)^2}{2\beta^2}} \mathbf{1}_{\mathbb{R}_+^*}(t),$$

the same for $\bar{W}_i'(t)$ *and* $\bar{L}_i'(t)$ *with* α_i *and* β_i *substituted by* α_i' *and* β_i'.

Following [CTE], we take:

$$\alpha_1 = 1/2216^{1.5} \simeq 9.59 \times 10^{-6}; \beta_1 = 1.5; \alpha_1' = 1/11284^2 \simeq 7.85 \times 10^{-9}; \beta_1' = 2;$$
$$\alpha_2 = 4.5; \beta_2 = 0.5; \alpha_2' = 1/2000^{1.2} \simeq 1.09 \times 10^{-4}; \beta_2' = 1.2$$

with

$$m_1 \simeq 2000; cv_1 \simeq 0.68; m_1' \simeq 10^4; cv_1' \simeq 0.52;$$
$$m_2 \simeq 102; cv_2 \simeq 0.53; m_2' \simeq 1881; cv_2' \simeq 0.84$$

Bounds for $P_t(1,j)$ $(j = 1,..,4)$ *as well as numerical results from [CTE] and asymptotic values are plotted in Figure 2 with* $0 \le t \le 2 \times 10^5$, $h = 4$ *for* $j = 1$, $h = \frac{2}{3}$ *for* $j = 2$ *and* $h = 40$ *for* $j = 3$ *and* 4. *We can see in such a figure that our results seem to more or less coïncide with those from [CTE].*

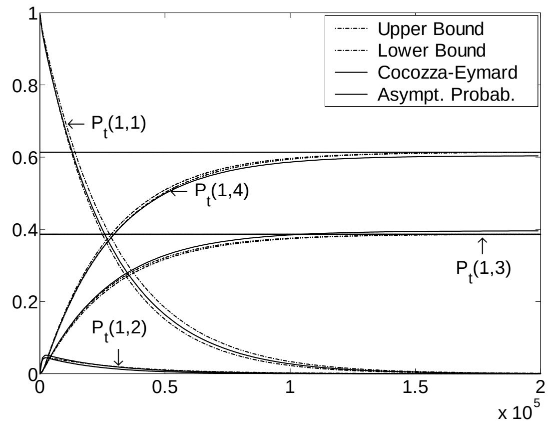

Figure 2. Example 10, $P_t(1,j)$ *for* $j = 1..4$.

In order to better compare the results, we now give in Table 2 the results from [CET] and our bounds for $P_t(1,3)$, for different values of h and t.

t	h	*Lower bound*	*Upper bound*	t	h	*Lower bound*	*Upper bound*
	80	0	4×10^{-4}		200	0.100 37	0.120 85
	40	2.2×10^{-5}	1.9×10^{-4}		100	0.105 22	0.115 50
80	20	3.4×10^{-5}	1.1×10^{-4}	10^4	40	0.107 73	0.111 82
	10	4.1×10^{-5}	7.5×10^{-5}		20	0.108 64	0.110 68
	5	4.46×10^{-5}	6.08×10^{-5}	*[CTE]*	-	0.112 81	
	80	1×10^{-4}	8×10^{-4}		200	0.307 44	0.330 49
	40	2×10^{-4}	5×10^{-4}		100	0.313 81	0.325 38
160	20	2.2×10^{-4}	3.7×10^{-4}	5×10^4	40	0.316 98	0.321 61
	10	2.4×10^{-4}	3.1×10^{-4}		20	0.318 12	0.320 44
	5	2.48×10^{-4}	2.84×10^{-4}	*[CTE]*	-	0.328	
	200	5.5×10^{-3}	1.1×10^{-2}		400	0.363 67	0.380 12
	100	6.4×10^{-3}	8.9×10^{-3}		200	0.370 57	0.378 39
10^3	40	6.8×10^{-3}	7.8×10^{-3}	10^5	100	0.373 01	0.376 90
	20	6.96×10^{-3}	7.45×10^{-3}		40	0.374 15	0.375 72
[CTE]	-	0.007 213		*[CTE]*	-	0.384 61	
	200	5.04×10^{-2}	0.063728		400	0.385 00	0.386 25
	100	5.33×10^{-2}	0.059998		200	0.385 70	0.386 19
5×10^3	40	5.48×10^{-2}	5.75×10^{-2}	1.99×10^5	100	0.385 89	0.386 12
	20	5.537×10^{-2}	5.669×10^{-2}		40	0.385 96	0.386 06
[CTE]	-	0.057 54		*[CTE]*	-	0.395 72	

Table 2. Example 10, $P_t(1,3)$ with respect of h and comparison with the results from [CTE]

Here again, the smaller the discretization step h , the tighter the bounds are. Also, for h fixed, the error first increases with t and then decreases.

As for the comparison between the method from [CTE] and the present one, one can look at the asymptotic value: we know that $P_t(1,3)$ is increasing with respect of t (see Figure 2) and that $\lim_{t\to+\infty} P_t(1,3) \simeq 0.386\ 365$ (easy computation using standard method). In Table 2, we can see that the value from [CTE] for $P_t(1,3)$ and $t = 1.99 \times 10^5$ is slightly too big (0.395 72) whereas our bounds (0.385 96 and 0.386 06) are coherent with the asymptotic value. A *similar remark is valid for $P_t(1,4)$ with $P_t(1,4)$ slightly too small in [CTE] for $t = 1.99 \times 10^5$ whereas our bounds are coherent.*

We can then conclude that, in this example, our bounds seem slightly more accurate than the results from [CTE] (which may however be improved using some smaller discretization step).

In case where the semi-Markov kernel is triangular, we finally provide a last numerical example to compare the bounds provided by Theorem 3 and by Proposition 4.

Example 11 *A semi-Markov unreparable system is considered with state space $E = \{1, ..., 5\}$, $U = \{1, 2, 3\}$, $D = \{4, 5\}$. We want to evaluate the unreliability of the system starting from 1, namely $F(t) = \mathbb{P}_1(T \leq t)$ where T is the time to failure of the system. The semi-Markov kernel is:*

$$q(\cdot, \cdot, dt)$$

$$= \begin{pmatrix} 0 & (1-\gamma)\, w_1(t)\, dt + \gamma\delta_0 & 0 & 0 & 0 \\ 0 & 0 & (1-\gamma)\, w_2(t)\, dt + \gamma\delta_0 & 0 & 0 \\ 0 & 0 & 0 & (1-\gamma)\, w_3(t)\, dt + \gamma\delta_0 & 0 \\ 0 & 0 & 0 & 0 & (1-\gamma)\, w_4(t)\, dt + \gamma\delta_0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with w_1, w_2, w_3, w_4 and γ the same as in Example 9.

Due to the triangular shape of the kernel, we here have: $F(t) = \mathbb{P}_1(X_t \geq 4) = f_w(t)$ with $w(i,t) = v(i)\, \mathbb{P}_i(T_1 \leq t)$ and $v(i) = \mathbf{1}_{\{i \geq 4\}}$, all $i \in E$ (see Proposition 4 and Remark 6). As v is non-decreasing and $q(\cdot, \cdot, dt)$ is upper triangular, we may apply (9)*. The bounds for the unreliability are plotted in Figure 5 for $h = 5$ as provided by* (14) *(bounds 1) and by* (9) *(bounds 2).*

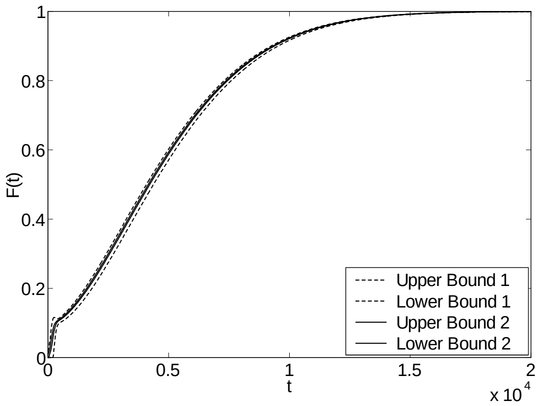

Figure 5. Example 11, $F(t)$ by the two methods, $h = 20$

We can see that, as expected, the bounds provided by (9) *are tighter than those provided by* (14)*. Beside, the computations are quicker for* (9) *than for* (14)*. Then, it is better to use* (9) *than* (14) *when possible.*

6 Conclusion

We have proposed, in this paper, easily computable bounds for different time-dependent quantities for semi-markovian system. The idea used here, namely approximating continuous-time SMP by discrete-time SMP is not new. However, contrary to most of previous works (apart from [Elkins & Wortman (2001)], to our knowledge), the present construction of the approximating processes has allowed us to control the error between the goal quantities and their approximations. The theoretical results have been tested on a few numerical examples, showing the accuracy of the method.

We have focused here in solutions of Markov renewal equations. Note however that the bounding method might be generalized and that for any function Φ non-decreasing e.g. with respect to all T_n and such that the expectations exist, we would have:

$$\begin{aligned}\mathbb{E}\left(\phi\left((Y_n)_{n\in\mathbb{N}}, \left(T_n^{(h)}\right)_{n\in\mathbb{N}}\right)\right) &\leq \mathbb{E}\left(\phi\left((Y_n)_{n\in\mathbb{N}}, (T_n)_{n\in\mathbb{N}}\right)\right)\\ &\leq \mathbb{E}\left(\phi\left((Y_n)_{n\in\mathbb{N}}, \left(T_n^{(h+)}\right)_{n\in\mathbb{N}}\right)\right)\end{aligned}$$

where $T_n^{(h)} = \sum_{i=0}^{n-1} (T_{i+1} - T_i)^h = \sum_{i=0}^{n-1} h \left\lfloor \frac{T_{i+1}-T_i}{h} \right\rfloor$ and $T_n^{(h+)} = T_n^{(h)} + nh$.

More generally, one could also consider $\bar{h} = (h_i)_{i \in E}$ and the semi-Markov kernels $q^{\bar{h}}(i,.,dt)$ and $q^{\bar{h}+}(i,.,dt)$ (both supported by $h_i\mathbb{N}$) with:

$$q^{\bar{h}}(i,j,kh_i) = \mathbb{P}_i(Y_1 = j, kh_i \leq T_1 < (k+1)h_i)$$
$$q^{\bar{h}+}(i,j,kh_i) = \mathbf{1}_{\{k \geq 1\}} q^{\bar{h}}(i,j,(k-1)h_i)$$

Here again, the bounding would remain true. However, more general ϕ than in the present paper and/or non constant step size $\bar{h}$ would lead to more complicated computations of the approximate quantities, which only require here a few lines of implementation.

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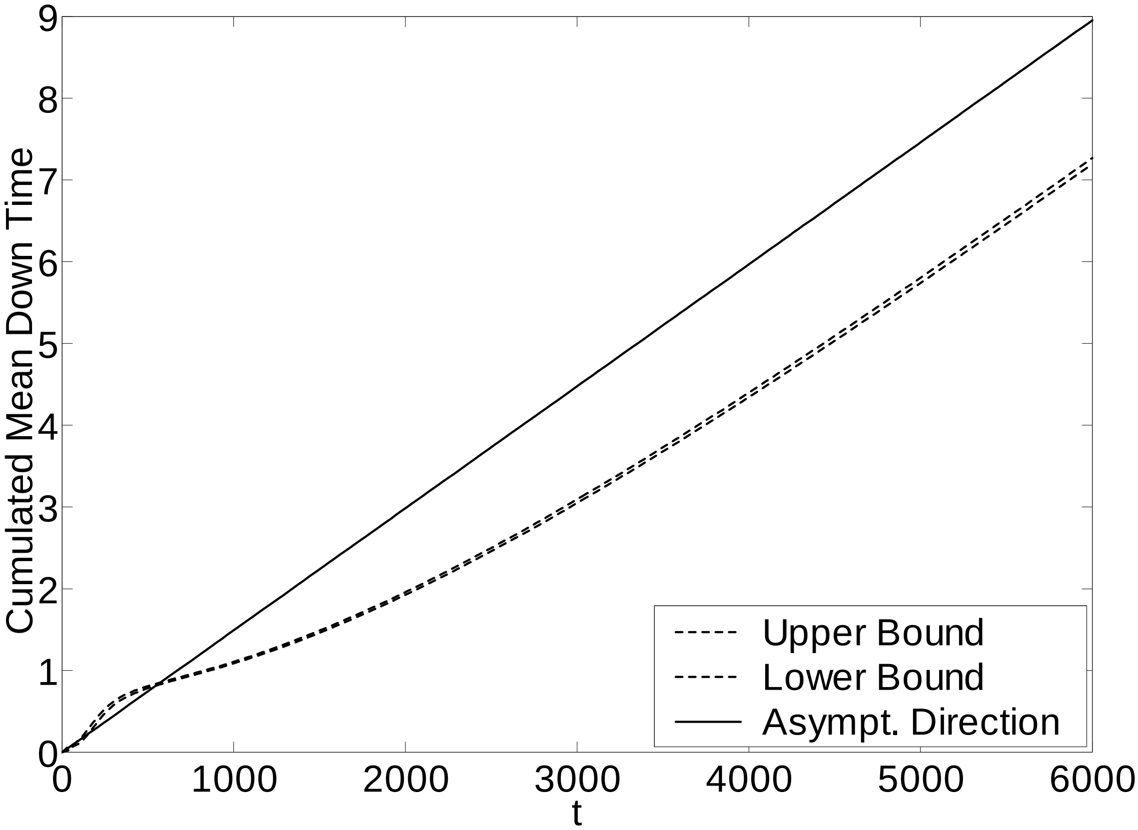

Figure 1. Example 9, Cumulated Mean Down Time on $[0, t]$, $h = 6$.

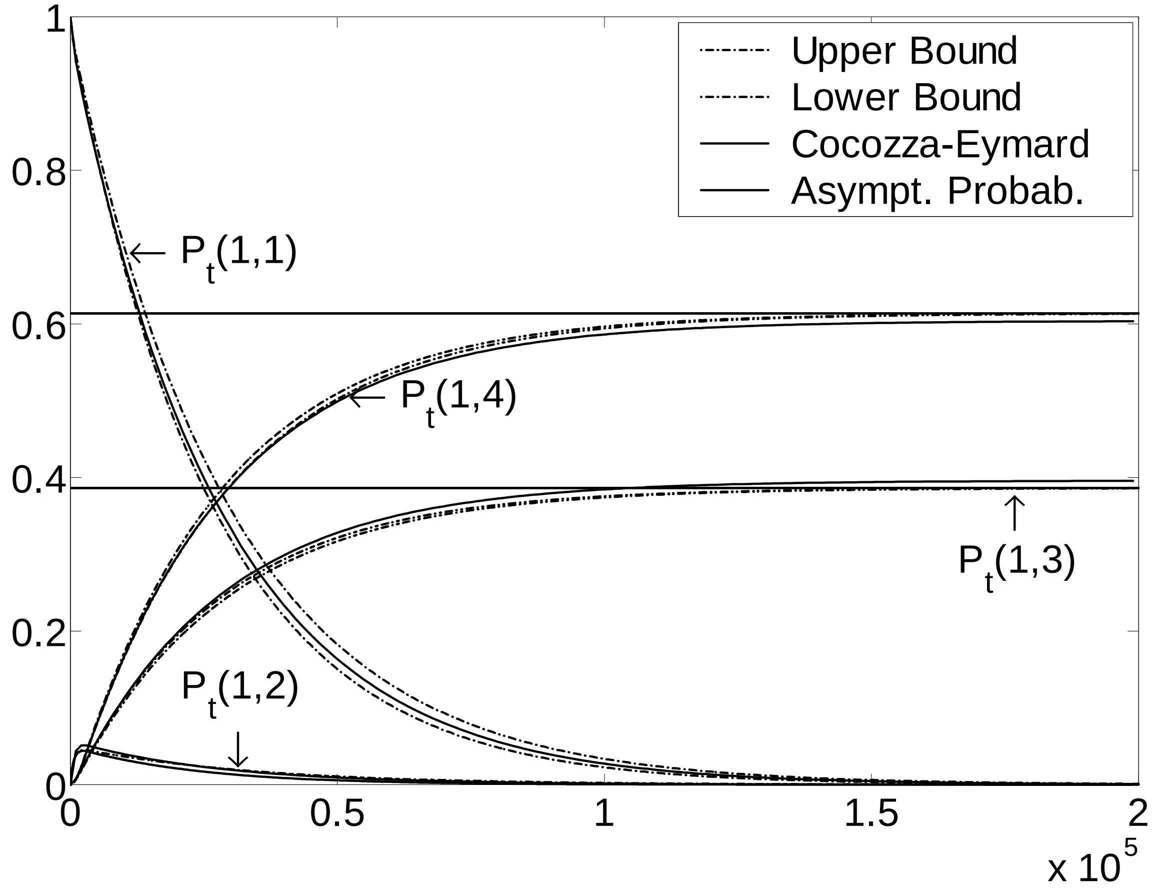

Figure 2. Example 10, $P_t\,(1, j)$ for $j = 1..4$.

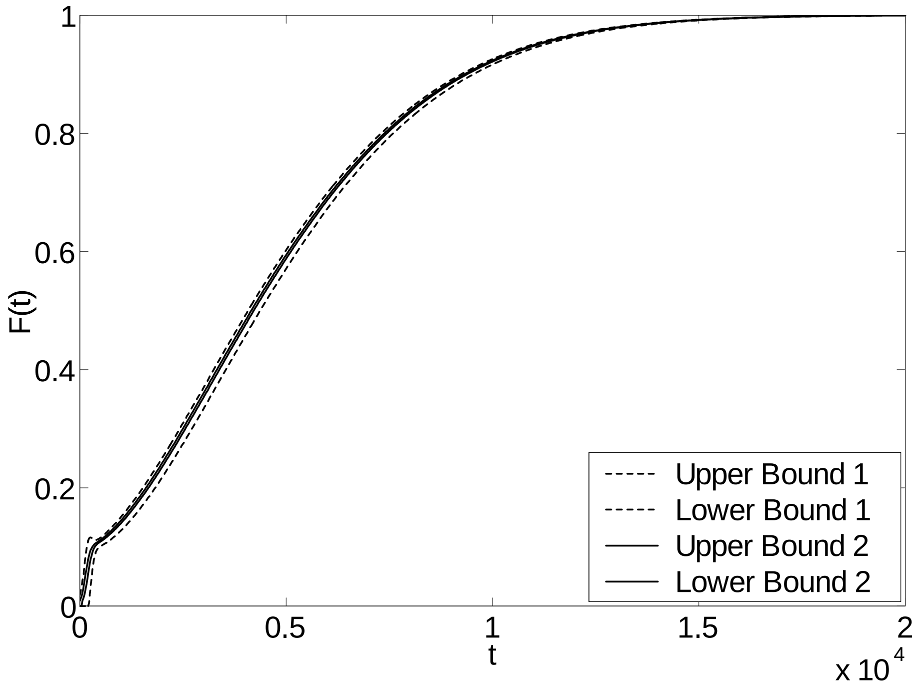

Figure 3. Example 11, $F\left(t\right)$ by the two methods, $h=20$